Shapes of numbers and the geometric Manin's conjecture

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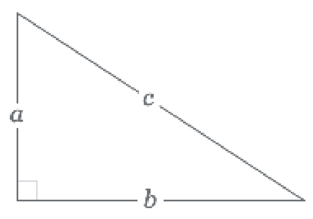
Introduction

Fermat, in early 17th century, asked: if the sides of a right-angled triangle are numbers a, b, c coming from $\{1, 2, 3, \cdots\}$, can its area *ever* be a square number?

A seemingly elementary question at the intersection of geometry and number theory, but the problem is unsolved to this day!



Fermat's story illustrates well what mathematicians have known all along: (i) there are open problems and conjectures galore in number theory (many of them unsolved for centuries, others decades old) and (ii) geometry and the theory of numbers are intricately and inextricably related. The renowned ***Manin's conjecture*** is one such: it relates the number of solutions of (a cluster of) equations, to the number of lines and planes you can draw on the *graph* of the equation(s).

Objectives

There are some special graphs of equations that give rise to geometric shapes called *Fano varieties*; they are the building blocks of graphs of more complicated equations. Amongst those *Fano varieties* are shapes with additional symmetries, called *almost-solv varieties*. My objectives for this summer were:

Aim 1: *Build an algorithm to construct suitable* ***binary algebraic operations*** *on the solution-sets of those equations whose graphs are general Fano varieties. Show that these operations, when well-defined, imply Manin's conjecture.*

Aim 2: *Run the aforementioned algorithm on* ***almost-solv varieties*** *by exploiting their symmetries, and prove Manin's conjecture for those cases.*

Methodology

The main tool in forming a bridge between geometry and number theory is the program of **shapes of numbers**. The program takes, as input, certain numerical parameters associated with geometric shapes (called the *cohomology* of those shapes, they come in the form of sequences of numbers like $\{n_1, n_2, \cdots\}$), and gives, as output, an approximate count on the number of solutions of equations in remainders on dividing by a prime.

There is no general strategy for computing cohomology of geometric shapes: not even for special shapes like *Fano varieties*. But in the case of *almost-solv varieties* there are certain symmetries which I can, and do, exploit, to obtain their cohomology, and subsequently run the program of shapes of numbers.

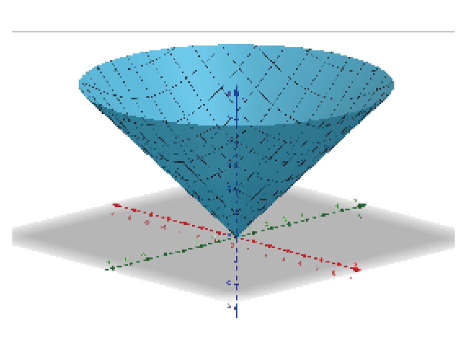
An example to support the results

The equation $z^2 = x^2 + y^2$ has six solutions when one allows x, y, z to be numbers that are remainders on dividing by 5: plug in $0, \cdots, 4$ as possible values of x, y, z, and record those that satisfy the equation– it takes a second using a computer.

But 5 is a small prime, what about the prime $2^{82,589,933} - 1$?

We cannot possibly hope to plug, in place of each of x, y, z, all the remainders on dividing by $2^{82,589,933} - 1$,i.e., all possible numbers from $\{0, 1, 2, 3, \cdots, 2^{82,589,933} - 2\}$, in the equation $x^2 + y^2 = z^2$, and check with a computer which of them produce solutions– that would be madness!

Figure 1. Graph of $z^2 = x^2 + y^2$: a cone



$2^{82,589,933} - 1$ is the largest known prime number as of December 2023; it has 24862048 digits. It was found in 2018 using the software GIMPS. The largest known prime number before this was 524287, found 144 years ago. In other words, modern technology would probably take a few hundred years to count all the solutions of $x^2 + y^2 = z^2$ in remainders of $2^{82,589,933} - 1$.

Here comes the program of *shapes of numbers* to the rescue. The graph of $z^2 = x^2 + y^2$ in Figure 1 is a geometric shape: it is a cone (like an empty icecream cone). To 'see' its cohomology, observe: (i) you can draw straight lines on the cone (like rays from its vertex): lines are 1-dimensional, (ii) you can draw points on the cone: points are 0-dimensional, (iii) you cannot draw a flat plane or anything of bigger dimension on the cone. In effect, the cohomology of Figure 1 is the sequence of numbers $\{1, 0\}$, from the dimensions of shapes you can draw on the cone. So $\{1, 0\}$ is the input to the program of shapes of numbers; and $p^1 + p^0 = p + 1$, the output, is the number of solutions of $z^2 = x^2 + y^2$ in remainders on dividing by p. In the case when $p = 5$ we recover our previous count: that there are 6 solutions. And if you put $p = 2^{82,589,933} - 1$ you get that the number of solutions of $z^2 = x^2 + y^2$ in remainders of $(2^{82,589,933} - 1)$, is $2^{82,589,933}$.

What the world's most powerful computer cannot do, shapes of numbers can!

Results

Main Theorem: *Manin's conjecture is verified for almost-solv varieties.*

Previously, as a first instalment of my ongoing project, I showed that *toric varieties*, which are special types *almost-solv varieties* with strong additional symmetries, satisfy Manin's conjecture ([B]). This summer, the computations for toric varieties have been successfully extrapolated to the case of almost-solv varieties using foundational matters from my earlier work in [B24].

Comparison with existing results

What distinguishes my approach to all the previous attempts at resolving Manin's conjecture is how I use geometry, as opposed to the traditional techniques form number theory. In fact it was T. Browning, one of the leading experts in Manin's conjecture, who raised the question whether the conjecture can be addressed using geometric tools (see [BV14], [BS20]). I not only provide an answer in affirmative, I also provide a concrete algorithm that can be run to verify Manin's conjecture, should certain parameters on the symmetries of the geometric shapes like Fano varieties can be computed.

Future of this research

My studying a decades-old number theory conjecture using modern tools from *geometry* has started carving a path in the newly emerging and rapidly developing field that goes by the name of *arithmetic topology*: its centrepiece the program of shapes of numbers.

Furthermore, the underpinnings of the previously mentioned binary algebraic operations, parts of which I wrote in [B24], have already started being utilised by other mathematicians to produce new results on *configuration spaces*, an object of *combinatorial geometry* and of great interest to topologists, geometers and algebraists alike (see [RW]).

References

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